

ON GREEN'S FUNCTION OF AN n -POINT BOUNDARY VALUE PROBLEM

BY

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ABSTRACT. The Green's function $g_n(x, s)$ for an n -point boundary value problem, $y^{(n)}(x)=0$, $y(a_1) = y(a_2) = \dots = y(a_n) = 0$ is explicitly given. As a tool for discussing $\text{sgn } g_n(x, s)$ on the square $[a_1, a_n] \times [a_1, a_n]$, some results about polynomials with coefficients as symmetric functions of a 's are obtained. It is shown that

$$\int_{a_1}^{a_n} |g_n(x, s)| ds$$

is a suitable polynomial in x . Applications to n -point boundary value problems and lower bounds for a_m ($m \geq n$) are included.

1. Introduction. Beesack [1] considered the boundary value problem

$$(1.1) \quad \begin{aligned} y^{(n)}(x) &= 0, \\ y(a_i) &= y'(a_i) = \dots = y^{(k_i)}(a_i) = 0 \quad (1 \leq i \leq r), \end{aligned}$$

where $a_1 < a_2 < \dots < a_r$, $0 \leq k_i$, $k_1 + k_2 + \dots + k_r = n - r$. For the Green's function $g_n(x, s)$ he proved that

$$(1.2) \quad |g_n(x, s)| \leq \frac{\prod_{i=1}^r |x - a_i|^{k_i+1}}{(n-1)!(a_r - a_1)}.$$

In [2] Nehari gave a short proof of the same when $r = n$. Since in relation to multipoint boundary value problem as in [1], $\int_{a_1}^{a_r} |g_n(x, s)| ds$ appears (see 3.5 there), the natural question is to consider alternately this function.

In this paper, we consider (1.1) when $r = n$ first. In §2, we give the Green's function $g_n(x, s)$ explicitly and alternately exhibit it in a form which yields conclusions as to the sign of $g_n(x, s)$. In §3, the results about $\text{sgn } g_n(x, s)$ and the identity

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$$(1.3) \quad \int_{a_1}^{a_n} |g_n(x, s)| ds = \frac{1}{n!} (x - a_1)(a_n - x) \prod_{i=2}^{n-1} |x - a_i|$$

are obtained. (There are a few auxiliary results given as lemmas which may be of some independent interest!) Applications to n -point boundary value problems and lower bounds for the m th zero of solutions form the contents of §4.

2. **The Green's function.** Throughout, n denotes a fixed natural number greater than 2. Let k be a natural number such that $2 < k \leq n$. Consider the boundary value problem

$$(2.1) \quad \begin{aligned} y^{(k)}(x) &= 0, \\ y(a_1) &= y(a_2) = \dots = y(a_{k-1}) = y(a_n) = 0, \end{aligned}$$

where $a_1 < a_2 < \dots < a_{k-1} < a_n$.

Theorem 2.1. *The Green's function $g_k(x, s)$ for (2.1) is given by*

$$(2.2) \quad \begin{aligned} & (k-1)!g_k(x, s) \\ &= \left(\prod_{i=1}^{k-1} \frac{x - a_i}{a_n - a_i} \right) (a_n - s)^{k-1}, \quad x \leq s, \quad a_{k-1} \leq s; \\ &= \left(\prod_{i=1}^{k-1} \frac{x - a_i}{a_n - a_i} \right) (a_n - s)^{k-1} \\ &\quad + \sum_{j=2}^{r+1} (-1)^j \left(\prod_{\substack{i=1 \\ (i \neq k-j+1)}}^{k-1} \frac{x - a_i}{|a_{k-j+1} - a_i|} \right) \frac{a_n - x}{a_n - a_{k-j+1}} (a_{k-j+1} - s)^{k-1} \\ &\quad (\equiv g_k^r(x, s)), \quad x \leq s, \quad a_{k-r-1} \leq s \leq a_{k-r} \quad (r = 1, \dots, k-3); \end{aligned}$$

$$\begin{aligned} &= \left(\prod_{i=2}^{k-1} \frac{x - a_i}{a_i - a_1} \right) \frac{a_n - x}{a_n - a_1} (s - a_1)^{k-1} + (-1)^{k-1} (s - x)^{k-1}, \quad x \leq s, \quad s \leq a_2; \\ &= \left(\prod_{i=1}^{k-1} \frac{x - a_i}{a_n - a_i} \right) (a_n - s)^{k-1} - (x - s)^{k-1}, \quad a_{k-1} \leq s \leq x; \\ &= g_k^r(x, s) - (x - s)^{k-1}, \quad s \leq x, \quad a_{k-r-1} \leq s \leq a_{k-r} \quad (r = 1, \dots, k-3); \\ &= \left(\prod_{i=2}^{k-1} \frac{x - a_i}{a_i - a_1} \right) \frac{a_n - x}{a_n - a_1} (s - a_1)^{k-1}, \quad s \leq x, \quad s \leq a_2; \end{aligned}$$

where a product for empty set of indices is interpreted as 1.

Remark. Here, as well as in the following, r ranging over a vacuous set of indices means the collapse of regions of the form $x \leq s$ ($x \geq s$), $a_{k-r-1} \leq s \leq a_{k-r}$.

Proof. Starting with the Green's function for $y''(x) = 0$, $y(a_1) = y(a_2) = 0$, namely

$$g_2(x, s) = \begin{cases} (x - a_1)(a_n - s)/(a_n - a_1), & x \leq s; \\ (a_n - x)(s - a_1)/(a_n - a_1), & s \leq x; \end{cases}$$

and the relation in [1],

$$(2.3) \quad g_{m+1}(x, s) = \frac{1}{m} \left\{ (x - s)g_m(x, s) - (a_m - s)g_m(a_m, s) \left(\prod_{i=1}^{m-1} \frac{x - a_i}{a_m - a_i} \right) \frac{a_n - x}{a_n - a_m} \right\},$$

it is easily checked that

$$2!g_3(x, s) = \begin{cases} \frac{(x - a_1)(x - a_2)}{(a_n - a_1)(a_n - a_2)} (a_n - s)^2, & x \leq s, a_2 \leq s; \\ \frac{(x - a_2)(a_n - x)}{(a_2 - a_1)(a_n - a_1)} (s - a_1)^2 + (s - x)^2, & x \leq s, s \leq a_2; \\ \frac{(x - a_1)(x - a_2)}{(a_n - a_1)(a_n - a_2)} (a_n - s)^2 - (x - s)^2, & s \leq x, a_2 \leq s; \\ \frac{(x - a_2)(a_n - x)}{(a_2 - a_1)(a_n - a_1)} (s - a_1)^2, & s \leq x, s \leq a_2. \end{cases}$$

Thus (2.2) is valid for $k = 3$. Again, assuming that (2.2) holds when $k = m$, if $x \leq s$, $a_m \leq s$, then (2.3) gives

$$\begin{aligned} m!g_{m+1}(x, s) &= \left(\prod_{i=1}^{m-1} \frac{x - a_i}{a_n - a_i} \right) \left\{ (x - s) - (a_m - s) \frac{a_n - x}{a_n - a_m} \right\} (a_n - s)^{m-1} \\ &= \left(\prod_{i=1}^m \frac{x - a_i}{a_n - a_i} \right) (a_n - s)^m. \end{aligned}$$

Similarly, if $x \leq s$, $a_{m-r} \leq s \leq a_{m-r+1}$, where $r = 2, \dots, (m+1) - 3$, (2.3) yields

$$\begin{aligned}
& m!g_{m+1}(x, s) \\
&= \left((x-s) - (a_m - s) \frac{a_n - x}{a_n - a_m} \right) \left(\prod_{i=1}^{m-1} \frac{x - a_i}{a_n - a_i} \right) (a_n - s)^{m-1} + \left(\prod_{i=1}^{m-1} \frac{x - a_i}{a_m - a_i} \right) \frac{a_n - x}{a_n - a_m} (a_m - s)^m \\
&\quad + \sum_{j=2}^r (-1)^j \left(\prod_{\substack{i=1 \\ (i \neq m-j+1)}}^{m-1} \frac{x - a_i}{|a_{m-j+1} - a_i|} \right) \frac{a_n - x}{a_n - a_{m-j+1}} \\
&\quad \cdot (a_{m-j+1} - s)^{m-1} \left((x-s) - (a_m - s) \frac{x - a_{m-j+1}}{a_m - a_{m-j+1}} \right) \\
&= \left(\prod_{i=1}^m \frac{x - a_i}{a_n - a_i} \right) (a_n - s)^m + \sum_{j=1}^r (-1)^{j+1} \left(\prod_{\substack{i=1 \\ (i \neq m-j+1)}}^m \frac{x - a_i}{|a_{m-j+1} - a_i|} \right) \frac{a_n - x}{a_n - a_{m-j+1}} (a_{m-j+1} - s)^m.
\end{aligned}$$

A similar computation using the first expression of (2.2) with $k = m$ gives the result for $x \leq s$, $a_{m-1} \leq s \leq a_m$. Moreover, if $x \leq s \leq a_2$, then

$$\begin{aligned}
& m!g_{m+1}(x, s) \\
&= (-1)^m (s - x)^m + \left(\prod_{i=2}^{m-1} \frac{x - a_i}{a_i - a_1} \right) \frac{a_n - x}{a_n - a_1} (s - a_1)^{m-1} \left((x-s) - (a_m - s) \frac{x - a_1}{a_m - a_1} \right) \\
&= \left(\prod_{i=2}^m \frac{x - a_i}{a_i - a_1} \right) \frac{a_n - x}{a_n - a_1} (s - a_1)^m + (-1)^m (s - x)^m.
\end{aligned}$$

This completes the induction argument for the triangle $x \leq s$. On the same lines the region $a_1 \leq s \leq x \leq a_n$ can be handled. Hence the conclusion.

Corollary 2.2. *Alternatively, if $s \leq x$ and $a_{k-r-1} \leq s \leq a_{k-r}$ where $r = 1, \dots, k-3$, we have*

$$\begin{aligned}
(2.4) \quad (k-1)!g_k(x, s) &= (a_n - x) \sum_{l=0}^{k-r-2} (x-s)^l \left(\prod_{i=l+2}^{k-1} (x - a_i) \right) (s - a_{l+1}) \\
&\quad \cdot \left\{ \frac{(a_n - s)^{k-l-2}}{\prod_{i=l+1}^{k-1} (a_n - a_i)} + \sum_{j=2}^{r+1} (-1)^{j+1} \frac{(a_{k-j+1} - s)^{k-l-2}}{\prod_{i=l+1}^{k-1} (i \neq k-j+1) |a_{k-j+1} - a_i|} \frac{1}{a_n - a_{k-j+1}} \right\}.
\end{aligned}$$

Proof. First observe that $(x - a_i)(a_m - s) = (a_m - a_i)(x - s) + (s - a_i)(a_m - x)$, $i \neq m$. Applying this first with $i = 1$ and $m = n$ or $k - j + 1$ —note that m is always different from 1 —, we get

$$\begin{aligned}
g_k^r(x, s) - (x-s)^{k-1} &= (a_n - x) \left[\left(\prod_{i=2}^{k-1} (x - a_i) \right) (s - a_1) \right. \\
&\quad \cdot \left. \left\{ \frac{(a_n - s)^{k-2}}{\prod_{i=1}^{k-1} (a_n - a_i)} + \sum_{j=2}^{r+1} (-1)^{j+1} \frac{(a_{k-j+1} - s)^{k-2}}{\prod_{i=1}^{k-1} (i \neq k-j+1) |a_{k-j+1} - a_i|} \frac{1}{a_n - a_{k-j+1}} \right\} \right] \\
&\quad + (x-s) \left[\left(\prod_{i=2}^{k-1} \frac{x - a_i}{a_n - a_i} \right) (a_n - s)^{k-2} - (x-s)^{k-2} \right. \\
&\quad \left. + \sum_{j=2}^{r+1} (-1)^j \left(\prod_{\substack{i=2 \\ (i \neq k-j+1)}}^{k-1} \frac{x - a_i}{|a_{k-j+1} - a_i|} \right) \frac{a_n - x}{a_n - a_{k-j+1}} (a_{k-j+1} - s)^{k-2} \right].
\end{aligned}$$

Repeating the above with $i = 2, \dots, k-r-1$ and $m = n$ or $k-j+1$ (once again $i \neq m$ always) on the last term each time, we finally obtain

$$\begin{aligned}
g_k^r(x, s) - (x-s)^{k-1} &= (a_n - x) \sum_{l=0}^{k-r-2} (x-s)^l \left(\prod_{i=l+2}^{k-1} (x - a_i) \right) (s - a_{l+1}) \\
&\quad \cdot \left\{ \frac{(a_n - s)^{k-l-2}}{\prod_{i=1}^{k-1} (a_n - a_i)} + \sum_{j=2}^{r+1} (-1)^{j+1} \frac{(a_{k-j+1} - s)^{k-l-2}}{\prod_{i=l+1}^{k-1} (i \neq k-j+1) |a_{k-j+1} - a_i|} \frac{1}{a_n - a_{k-j+1}} \right\} \\
&\quad + (x-s)^{k-r-1} \left[\left(\prod_{i=k-r}^{k-1} \frac{x - a_i}{a_n - a_i} \right) (a_n - s)^r - (x-s)^r \right. \\
&\quad \left. + \sum_{j=2}^{r+1} (-1)^j \left(\prod_{\substack{i=k-r \\ (i \neq k-j+1)}}^{k-1} \frac{x - a_i}{|a_{k-j+1} - a_i|} \right) \frac{a_n - x}{a_n - a_{k-j+1}} (a_{k-j+1} - s)^r \right].
\end{aligned}$$

Now (2.4) follows from the above in view of the fact that the factor multiplying $(x-s)^{k-r-1}$ is a polynomial of degree r in x and takes the value zero at $a_{k-r}, \dots, a_{k-1}$ and a_n .

3. We first give some auxiliary results in the form of lemmas.

Lemma 3.1. For each $r = (1, \dots, k-3)$ if ρ , a natural number, does not exceed r ; then

$$(3.1) \quad \frac{(a_n - s)^{\rho-1}}{\prod_{i=k-r}^{k-1} (a_n - a_i)} + \sum_{j=2}^{r+1} (-1)^{j+1} \frac{(a_{k-j+1} - s)^{\rho-1}}{\prod_{i=k-r}^{k-1} (i \neq k-j+1) |a_{k-j+1} - a_i|} \frac{1}{a_n - a_{k-j+1}} \equiv 0$$

and hence ($\equiv A_{r, \rho}^{(k)}(s)$)

$$(3.2) \quad \frac{(a_n - s)^\rho}{\prod_{i=k-r-1}^{k-1} (a_n - a_i)} + \sum_{j=2}^{r+1} (-1)^{j+1} \frac{(a_{k-j+1} - s)^\rho}{\prod_{i=k-r-1}^{k-1} (i \neq k-j+1) |a_{k-j+1} - a_i|} \frac{1}{a_n - a_{k-j+1}}$$

($\equiv B_{r, \rho}^{(k)}(s)$)

is divisible by $(a_{k-r-1} - s)^\rho$.

Proof. That (3.2) has $(a_{k-r-1} - s)^\rho$ as a factor is immediate when relations (3.1) are known to be true as seen from the identity

$$B_{r, \rho}^{(k)}(s) = (a_{k-r-1} - s)^\rho B_{r, 0}^{(k)} + \sum_{\sigma=0}^{\rho-1} (a_{k-r-1} - s)^\sigma A_{r, \rho-\sigma}^{(k)}(s)$$

obtained by successive use of

$$B_{r, \rho-\sigma}^{(k)}(s) = (a_{k-r-1} - s) B_{r, \rho-\sigma-1}^{(k)} + A_{r, \rho-\sigma}^{(k)}(s), \quad \sigma = 0, 1, \dots, \rho-1.$$

To establish (3.1) for arbitrary r ($\leq k-3$) and $\rho = 1$, it is sufficient to observe that the polynomial

$$1 + \sum_{j=2}^{r+1} (-1)^{j+1} \left(\prod_{\substack{i=k-r \\ (i \neq k-j+1)}}^{k-1} \frac{t - a_i}{|a_{k-j+1} - a_i|} \right)$$

(of degree $r-1$) vanishes at $a_{k-r}, \dots, a_{k-1}$ and hence identically. (This proves (3.1) when $r = 1$.) Now assume that (3.1) holds when r (> 1) is replaced by $r-1$ and for $\rho = 1, \dots, \sigma$ ($< r$). Then, if $\rho = \sigma + 1$, (3.1) follows from the identity

$$A_{r, \sigma+1}^{(k)}(s) = A_{r-1, \sigma}^{(k)}(s) + (a_{k-r} - s) A_{r, \sigma}^{(k)}(s), \quad 2 \leq r \leq k-3, 1 \leq \sigma < r.$$

This completes the proof by induction.

Remark. In the special case when $\rho = r$, (3.2) is a polynomial of degree r . In view of the above result, we shall write this polynomial as $C_r^{(k)}(a_{k-r-1} - s)^r$, where k is fixed and $C_r^{(k)}$ are constants for $r = 1, \dots, k-3$. In fact, $C_r^{(k)} = B_{r, 0}^{(k)}$ as can be easily checked.

Lemma 3.2. For each r ($= 1, \dots, k-3$), $C_r^{(k)}(a_{k-r-1} - s)^r$ is positive on (a_{k-r-1}, a_{k-r}) .

Proof. It is easy to check that for $r = 1$.

$$C_1^{(k)}(a_{k-2} - s) \equiv \frac{(s - a_{k-2})}{(a_n - a_{k-2})(a_{k-1} - a_{k-2})}$$

and thus, in addition to the conclusion, we have $C_1^{(k)} < 0$.

Now assume that for $r = \sigma - 1$, the conclusion is true and $\text{sgn } C_{\sigma-1}^{(k)} = (-1)^{\sigma-1}$. Then, if $r = \sigma$, we have

$$C_{\sigma}^{(k)}(a_{k-\sigma-1}-s)^{\sigma} \equiv (a_{k-\sigma-1}-s)^{\sigma-1} \cdot \left\{ \frac{a_n-s}{\prod_{i=k-\sigma-1}^{k-1}(a_n-a_i)} + \sum_{j=2}^{\sigma+1} (-1)^{j+1} \frac{a_{k-j+1}-s}{\prod_{i=k-\sigma-1}^{k-1} (i \neq k-j+1) |a_{k-j+1}-a_i|} \frac{1}{a_n-a_{k-j+1}} \right\} \\ (\equiv (a_{k-\sigma-1}-s)^{\sigma-1} H(s)),$$

in view of

$$B_{\sigma, \sigma}^{(k)}(s) = (a_{k-\sigma-1}-s)^{\sigma-1} B_{\sigma, 1}^{(k)}(s).$$

Note that

$$H(a_{k-\sigma-1}) = A_{\sigma, 1}^{(k)}(a_{k-\sigma-1}) = 0.$$

Moreover, the sign of $H(s)$ is constant on $(a_{k-\sigma-1}, a_{k-\sigma})$ and is that of $H(a_{k-\sigma})$, namely

$$\left(\prod_{\substack{i=k-\sigma-1 \\ (i \neq k-\sigma)}}^{k-1} (a_n - a_i) \right)^{-1} + \sum_{j=2}^{\sigma} (-1)^{j+1} \left(\prod_{\substack{i=k-\sigma-1 \\ (i \neq k-j+1, k-\sigma)}}^{k-1} |a_{k-j+1} - a_i| \right)^{-1} (a_n - a_{k-j+1})^{-1}$$

which is $\text{sgn } q(a_n)$, where

$$q(t) = 1 + \sum_{j=2}^{\sigma} (-1)^{j+1} \frac{t - a_{k-\sigma-1}}{a_{k-j+1} - a_{k-\sigma-1}} \left(\prod_{\substack{i=k-\sigma+1 \\ (i \neq k-j+1)}}^{k-1} \frac{t - a_i}{|a_{k-j+1} - a_i|} \right).$$

Also, $\text{sgn } C_{\sigma-1}^{(k)} = \text{sgn } p(a_n)$, where

$$p(t) = 1 + \sum_{j=2}^{\sigma} (-1)^{j+1} \prod_{\substack{i=k-\sigma \\ (i \neq k-j+1)}}^{k-1} \frac{t - a_i}{|a_{k-j+1} - a_i|}.$$

In view of the facts that both polynomials $p(t)$ and $q(t)$ are of degree $\sigma-1$, have the same zeros $a_{k-\sigma+l}$ ($l = 1, 2, \dots, \sigma-1$), and $p(a_{k-\sigma}) = q(a_{k-\sigma-1}) = 1$, it follows that $\text{sgn } H(a_{k-\sigma}) = \text{sgn } C_{\sigma-1}^{(k)}$. Thus, $C_{\sigma}^{(k)}(a_{k-\sigma-1}-s)^{\sigma}$ is positive in $(a_{k-\sigma-1}, a_{k-\sigma})$ and $\text{sgn } C_{\sigma}^{(k)} = (-1)^{\sigma}$. This completes the proof.

Lemma 3.3. For all integers k, r, m such that $4 \leq k (< n)$, $1 \leq r \leq k-3$, and $r \leq m \leq k-2$,

$$(3.3) \quad A(k, r, m, s) \equiv \frac{(a_n - s)^m}{\prod_{i=k-m-1}^{k-1} (a_n - a_i)} \\ + \sum_{j=2}^{r+1} (-1)^{j+1} \frac{(a_{k-j+1} - s)^m}{\prod_{i=k-m-1}^{k-1} (i \neq k-j+1) |a_{k-j+1} - a_i|} \frac{1}{a_n - a_{k-j+1}}$$

is nonnegative on $[a_{k-r-1}, a_{k-r}]$.

Proof. First observe that the assertion follows from Lemma 3.2 if $m = r$ and k arbitrary, admissible. Also, if $r = 1$ and m, k admissible, then the identity

$$(3.4) \quad \left(\frac{a_n - s}{a_n - a_{k-m-2}} - \frac{a_{k-1} - s}{a_{k-1} - a_{k-m-2}} \right) \frac{(a_n - s)^m}{\prod_{i=k-m-1}^{k-1} (a_n - a_i)} \\ \equiv \frac{(a_n - s)^m}{\prod_{i=k-m-2}^{k-2} (a_n - a_i)} \frac{s - a_{k-m-2}}{a_{k-1} - a_{k-m-2}},$$

in view of $a_{k-1} \geq s \geq a_{k-2} > a_{k-m-2}$, implies

$$\frac{(a_n - s)^{m+1}}{\prod_{i=k-m-2}^{k-1} (a_n - a_i)} \geq \frac{a_{k-1} - s}{a_{k-1} - a_{k-m-2}} \frac{(a_n - s)^m}{\prod_{i=k-m-1}^{k-1} (a_n - a_i)}.$$

Thus $A(k, 1, m, s)$ is nonnegative by using induction on m .

Now we may assume $r \geq 2$ and thus admissible $k \geq 5$. Let the conclusion be true about $A(k, r, m, s)$ for admissible r, k . Note that in addition to (3.4) we have the identities

$$(3.5_j) \quad \left(\frac{a_{k-1} - s}{a_{k-1} - a_{k-m-2}} - \frac{a_{k-j+1} - s}{a_{k-j+1} - a_{k-m-2}} \right) \frac{(a_{k-j+1} - s)^m}{\prod_{i=k-m-1}^{k-1} (i \neq k-j+1) |a_{k-j+1} - a_i|} \\ \equiv \frac{(a_{k-j+1} - s)^m}{\prod_{i=k-m-2}^{k-2} (i \neq k-j+1) |a_{k-j+1} - a_i|} \frac{s - a_{k-m-2}}{a_{k-1} - a_{k-m-2}}, \\ j = 3, \dots, r+1.$$

Multiplying each (3.5_j) by $(-1)^j (a_n - a_{k-j+1})^{-1}$ and adding all to (3.4) we get

$$A(k, r, m+1, s) = \frac{a_{k-1} - s}{a_{k-1} - a_{k-m-2}} A(k, r, m, s) + \frac{s - a_{k-m-2}}{a_{k-1} - a_{k-m-2}} B,$$

$$B = A(k-1, r-1, m, s).$$

By induction hypothesis $A(k, r, m, s)$ as well as $A(k-1, r-1, m, s)$ are non-negative on $[a_{k-r-1}, a_{k-r}]$ in view of the admissibility of $k-1$ and $r-1$ in addition to that of k and r .

The following theorem is the main result which leads to (1.3).

Theorem 3.4. For $g_k(x, s)$ the following holds:

$$\operatorname{sgn} g_k(x, s) = \begin{cases} 1, & (x, s) \in [a_{k-1}, a_n] \times [a_1, a_n], \\ (-1)^r, & (x, s) \in [a_{k-r-1}, a_{k-r}] \times [a_1, a_n], \quad r = 1, \dots, k-2. \end{cases}$$

Proof. First we consider the triangle $a_1 \leq s \leq x \leq a_n$. The conclusion about $\operatorname{sgn} g_k(x, s)$ in this triangle is obvious from (2.2) when $s \leq a_2$, and immediate when $a_{k-1} \leq s \leq a_n$ since $(x - a_i)(a_n - s) \geq (a_n - a_i)(x - s)$ for $i = 1, \dots, k-1$. Also, if for $r = 1, \dots, k-3$, $s \in [a_{k-r-1}, a_{k-r}]$, then the assertion about $\operatorname{sgn} g_k(x, s)$ follows from (2.4) in view of Lemma 3.3, noting that $l+1 = k-m-1$ and that

$$\operatorname{sgn} \left(\prod_{i=k-m}^{k-1} (x - a_i) \right) = \begin{cases} 1, & x \in [a_{k-1}, a_n], \\ (-1)^r, & x \in [a_{k-r-1}, a_{k-r}]. \end{cases}$$

To discuss the triangle $a_1 \leq x \leq s \leq a_n$, we begin by observing that if $s \geq a_{k-1}$, then (2.2) at once gives the conclusion. For $s \leq a_{k-1}$, we use induction. First note that $g_3(x, s)$ has the asserted signs. Now assume that $g_m(x, s)$ has the asserted signs. Then (2.3) shows that if $x \in [a_{m-1}, a_m]$, $-\operatorname{sgn} g_{m+1}(x, s) \geq 0$. Also, noting that if $x \in [a_{m-r}, a_{m-r+1}]$ where $r = 2, \dots, m-1$, then $\operatorname{sgn} g_m(x, s) = \operatorname{sgn} (\prod_{i=1}^{m-1} (x - a_i))$, we have the desired conclusion for $k = m+1$.

This completes the proof.

Theorem 3.5. For any $k (\leq n)$ the following holds:

$$(3.6) \quad \int_{a_1}^{a_n} |g_k(x, s)| ds = \frac{1}{k!} (x - a_1)(a_n - x) \left(\prod_{i=2}^{k-1} |x - a_i| \right).$$

Proof. In view of Theorem 3.4,

$$(3.7) \quad \int_{a_1}^{a_n} |g_k(x, s)| ds = \left| \int_{a_1}^{a_n} g_k(x, s) ds \right|.$$

The value of the integral on the right-hand side by (2.2) is

$$\begin{aligned} & \frac{1}{k!} \left[\left(\prod_{i=2}^{k-1} \frac{x - a_i}{a_i - a_1} \right) \frac{a_n - x}{a_n - a_1} (a_2 - a_1)^k + \left(\prod_{i=1}^{k-1} \frac{x - a_i}{a_n - a_i} \right) (a_n - a_2)^k + (-1)^{k-1} (a_2 - x)^k \right] \\ & + \frac{1}{(k-1)!} \sum_{r=1}^{k-3} \int_{a_{k-r-1}}^{a_{k-r}} \left\{ \sum_{j=2}^{r+1} (-1)^j \left(\prod_{\substack{i=1 \\ (i \neq k-j+1)}}^{k-1} \frac{x - a_i}{|a_{k-j+1} - a_i|} \right) \frac{a_n - x}{a_n - a_{k-j+1}} (a_{k-j+1} - s)^{k-1} \right\} ds, \\ & \hspace{25em} x \in [a_1, a_2]; \\ & \frac{1}{k!} \left[\left(\prod_{i=2}^{k-1} \frac{x - a_i}{a_i - a_1} \right) \frac{a_n - x}{a_n - a_1} (a_2 - a_1)^k + \left(\prod_{i=1}^{k-1} \frac{x - a_i}{a_n - a_i} \right) (a_n - a_2)^k - (x - a_2)^k \right] \\ & + \frac{1}{(k-1)!} \sum_{r=1}^{k-3} \int_{a_{k-r-1}}^{a_{k-r}} \left\{ \sum_{j=2}^{r+1} (-1)^j \left(\prod_{\substack{i=1 \\ (i \neq k-j+1)}}^{k-1} \frac{x - a_i}{|a_{k-j+1} - a_i|} \right) \frac{a_n - x}{a_n - a_{k-j+1}} (a_{k-j+1} - s)^{k-1} \right\} ds, \\ & \hspace{15em} x \in [a_{k-1}, a_n] \text{ or } x \in [a_{k-l-1}, a_{k-l}], \quad l = 1, \dots, k-3. \end{aligned}$$

Thus, whatever $x \in [a_1, a_n]$,

$$\begin{aligned}
 & \int_{a_1}^{a_n} g_k(x, s) ds \\
 &= \frac{1}{k!} \left[\left(\prod_{i=2}^{k-1} \frac{x - a_i}{a_i - a_1} \right) \frac{a_n - x}{a_n - a_1} (a_2 - a_1)^k + \left(\prod_{i=2}^{k-1} \frac{x - a_i}{a_n - a_i} \right) (a_n - a_2)^k - (x - a_2)^k \right. \\
 & \quad \left. + \sum_{j=2}^{k-2} (-1)^j \left(\prod_{\substack{i=1 \\ (i \neq k-j+1)}}^{k-1} \frac{x - a_i}{|a_{k-j+1} - a_i|} \right) \frac{a_n - x}{a_n - a_{k-j+1}} (a_{k-j+1} - a_2)^k \right].
 \end{aligned}
 \tag{3.8}$$

It is easily seen that the expression in (3.8) is a polynomial (in x) of degree k which has zeros $a_1, \dots, a_{k-1}$ and a_n . Moreover, the coefficient of x^k is $-1/k$, hence the conclusion in (3.6).

4. Applications. In this section $k = n$. Thus, consider the ordinary differential equation

$$y^{(n)} + f(x, y, y', \dots, y^{(n-1)}) = 0, \tag{4.1}$$

where f is continuous on $[a_1, a_n] \times R^n$ and satisfies

$$|f(x, y, y', \dots, y^{(n-1)})| \leq K|y|. \tag{4.2}$$

(The above hypothesis is evidently no more restrictive than that of Beesack—see (3.2) in [1].)

The following lemma gives a bound which is better than (2.13) of [1] in situations which are not “highly pathological” (see Remark below).

Lemma 4.1. *Let $x \in [a_1, a_n]$. Then,*

$$\prod_{i=1}^n |x - a_i| \leq (n-1)^{n-1} \left(\frac{\delta}{2} \right)^n, \tag{4.3}$$

where $a_1 \leq a_2 \leq \dots \leq a_n$, $\delta = \max_{2 \leq i \leq n} (a_i - a_{i-1})$.

Proof. Let $x \in (a_r, a_{r+1})$, where $r \geq 1 < [(n+1)/2]$, the integral part of $(n+1)/2$. Then,

$$\begin{aligned}
& \left[\{(n-2r+1)(x-a_1)\}(x-a_2) \cdots (x-a_r) \prod_{i=r+1}^n (a_i-x) \right]^{1/n} \\
& \leq \frac{1}{n} \left(\sum_{i=r+1}^n (a_i-a_1) - \sum_{i=1}^r (a_i-a_1) \right) \\
& = \frac{1}{n} \left(\sum_{i=1}^{r-1} (n-2r+i)(a_{i+1}-a_i) + \sum_{i=r}^{n-1} (n-i)(a_{i+1}-a_i) \right) \\
& \leq \frac{n^2-n-2r(r-1)}{2n} \delta,
\end{aligned}$$

that is

$$\prod_{i=1}^n |x-a_i| \leq \frac{1}{n-2r+1} \left(\frac{\delta}{2n} \right)^n (n^2-n-2r(r-1))^n.$$

Similarly, if $[(n+1)/2] \leq r < n$, we have

$$\prod_{i=1}^n |x-a_i| \leq \frac{1}{n-2(n-r)+1} \left(\frac{\delta}{2n} \right)^n \{n(n-1)-2(n-r)(n-r-1)\}^n.$$

It is easy to check that

$$f(r) = \{n(n-1)-2r(r-1)\}^n / (n-2r+1)$$

is nonincreasing for $(1 \leq) r < [(n+1)/2]$ and $f(n-r)$ is nondecreasing for $([(n+1)/2] \leq) r < n$. The estimate (4.3) follows in view of $f(1) = f(n-1) = n^n(n-1)^{n-1}$.

Remark. The bound in (4.3) is better than Beesack's if and only if

$$\delta < 2(a_n - a_1)/n.$$

If $n \geq 3$, this is always the case when the a_i 's are equally spaced. In general, however, (2.13) of [1] gives a sort of best possible bound.

Theorem 4.2. Let the boundary value problem (4.1) and

$$(4.4) \quad y(a_1) = \cdots = y(a_n) = 0, \quad a_1 < a_2 < \cdots < a_n,$$

have a solution. Then,

$$(4.5) \quad K^{-1} < \begin{cases} \frac{(n-1)^{n-1}}{n!} \left(\frac{\delta}{2} \right)^n, & \text{if } \delta < \frac{2}{n}(a_n - a_1), \\ \frac{(n-1)^{n-1}}{n^n} \frac{(a_n - a_1)^n}{n!}, & \text{otherwise,} \end{cases}$$

where K and δ are as above.

Proof. (4.5) follows from the fact that $y(x)$ satisfies the integral equation

$$(4.6) \quad y(x) = \int_{a_1}^{a_n} g_n(x, s) f(s, y(s), \dots, y^{(n-1)}(s)) ds, \quad x \in [a_1, a_n],$$

and thus identifying x with a point where $|y(x)|$ attains its maximum, we have

$$(4.7) \quad 1 < K \int_{a_1}^{a_n} |g_n(x, s)| ds,$$

in view of (4.2).

Remark. The above result is an improvement on Beesack's necessary condition whenever the function $b(x)$ in his (3.2) is constant (of course, multiple zeros are not allowed). Apart from the case $b(x) \equiv K$, the two results are not comparable.

Next turning to the question of obtaining a lower bound for the m th zero of solutions of the linear differential equation

$$(4.8) \quad y^{(n)} + p(x)y = 0,$$

we state the following result:

Theorem 4.3. Let $p(x)$ in (4.8) be continuous and bounded on $[a, \infty)$. If $a_1 (\geq a) < a_2 < \dots < a_m$ are consecutive simple zeros of a solution of (4.8), then for $m > n$

$$(4.9) \quad a_m > a_1 + \left(\frac{(m-n+1)m!}{K} \left(\frac{n}{n-1} \right)^{n-1} \right)^{1/n},$$

where $|p(x)| \leq K$.

We omit the proof which is a straightforward adaptation of the above proof and of the proof of (3.15) in [1].

Remark. As in [1], if $m > 2n - 1$, in place of (4.9) we have the estimate

$$(4.10) \quad a_m > a_1 + (n/(n-1))((m-n)n!/K)^{1/n}.$$

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